

Q. 36 The value of $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is

(a) $9x^2(x+y)$

(b) $9y^2(x+y)$

(c) $3y^2(x+y)$

(d) $7x^2(x+y)$

Sol. (b) We have, $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$$= \begin{vmatrix} 3(x+y) & x+y & y \\ 3(x+y) & x & y \\ 3(x+y) & x+2y & -2y \end{vmatrix}$$

$[\because C_1 \rightarrow C_1 + C_2 + C_3 \text{ and } C_3 \rightarrow C_3 - C_2]$

$$= 3(x+y) \begin{vmatrix} 1 & (x+y) & y \\ 1 & x & y \\ 1 & (x+2y) & -2y \end{vmatrix}$$

$[\text{taking } 3(x+y) \text{ common from first column}]$

$$= 3(x+y) \begin{vmatrix} 0 & y & 0 \\ 1 & x & y \\ 1 & (x+2y) & -2y \end{vmatrix}$$

$[\because R_1 \rightarrow R_1 - R_2]$

Expanding along R_1 ,

$$= 3(x+y)[-y(-2y-y)]$$

$$= 3y^2 \cdot 3(x+y) = 9y^2(x+y)$$

Q. 37 If there are two values of a which makes determinant,

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86, \text{ then the sum of these number is}$$

(a) 4

(b) 5

(c) -4

(d) 9

Sol. (c) We have,

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$$

$$\Rightarrow 1(2a^2 + 4) - 2(-4a - 20) + 0 = 86$$

$[\text{expanding along first column}]$

$$\Rightarrow 2a^2 + 4 + 8a + 40 = 86$$

$$\Rightarrow 2a^2 + 8a + 44 - 86 = 0$$

$$\Rightarrow a^2 + 4a - 21 = 0$$

$$\Rightarrow a^2 + 7a - 3a - 21 = 0$$

$$\Rightarrow (a+7)(a-3) = 0$$

$$a = -7 \text{ and } 3$$

$$\therefore \text{Required sum} = -7 + 3 = -4$$

Q. 34 If A and B are invertible matrices, then which of the following is not correct?

(a) $\text{adj } A = |A| \cdot A^{-1}$

(b) $\det(A)^{-1} = [\det(A)]^{-1}$

(c) $(AB)^{-1} = B^{-1} A^{-1}$

(d) $(A + B)^{-1} = B^{-1} + A^{-1}$

Sol. (d) Since, A and B are invertible matrices. So, we can say that

$$(AB)^{-1} = B^{-1} A^{-1} \quad \dots(i)$$

Also, $A^{-1} = \frac{1}{|A|} (\text{adj } A)$

$$\Rightarrow \text{adj } A = |A| \cdot A^{-1} \quad \dots(ii)$$

Also, $\det(A)^{-1} = [\det(A)]^{-1}$

$$\Rightarrow \det(A)^{-1} = \frac{1}{[\det(A)]}$$

$$\Rightarrow \det(A) \cdot \det(A)^{-1} = 1 \quad \dots(iii)$$

which is true.

Again, $(A + B)^{-1} = \frac{1}{|(A + B)|} \text{adj } (A + B)$

$$\Rightarrow (A + B)^{-1} \neq B^{-1} + A^{-1} \quad \dots(iv)$$

So, only option (d) is incorrect.

Q. 35 If x , y and z are all different from zero and $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$,

then the value of $x^{-1} + y^{-1} + z^{-1}$ is

(a) xyz

(b) $x^{-1}y^{-1}z^{-1}$

(c) $-x - y - z$

(d) -1

Sol. (d) We have,

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$$

Applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$,

$$\Rightarrow \begin{vmatrix} x & 0 & 1 \\ 0 & y & 1 \\ -z & -z & 1+z \end{vmatrix} = 0$$

Expanding along R_1 ,

$$x[y(1+z) + z] - 0 + 1(yz) = 0$$

$$\Rightarrow x(y + yz + z) + yz = 0$$

$$\Rightarrow xy + xyz + xz + yz = 0$$

$$\Rightarrow \frac{xy}{xyz} + \frac{xyz}{xyz} + \frac{xz}{xyz} + \frac{yz}{xyz} = 0 \quad [\text{on dividing } (xyz) \text{ from both sides}]$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1 = 0$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -1$$

$$\therefore x^{-1} + y^{-1} + z^{-1} = -1$$

Q. 32 If $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then

(a) $f(a) = 0$

(b) $f(b) = 0$

(c) $f(0) = 0$

(d) $f(1) = 0$

Sol. (c) We have,

$$\begin{aligned} f(x) &= \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} \\ \Rightarrow f(a) &= \begin{vmatrix} 0 & 0 & a-b \\ 2a & 0 & a-c \\ a+b & a+c & 0 \end{vmatrix} \\ &= [(a-b) \{2a \cdot (a+c)\}] \neq 0 \\ \therefore f(b) &= \begin{vmatrix} 0 & b-a & 0 \\ b+a & 0 & b-c \\ 2b & b+c & 0 \end{vmatrix} \\ &= -(b-a) [2b(b-c)] \\ &= -2b(b-a)(b-c) \neq 0 \\ \therefore f(0) &= \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} \\ &= a(bc) - b(ac) \\ &= abc - abc = 0 \end{aligned}$$

Q. 33 If $A = \begin{vmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix}$, then A^{-1} exists, if

(a) $\lambda = 2$

(b) $\lambda \neq 2$

(c) $\lambda \neq -2$

(d) None of these

Sol. (d) We have,

$$A = \begin{vmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix}$$

Expanding along R_1 ,

$$|A| = 2(6-5) - \lambda(-5) - 3(-2) = 2 + 5\lambda + 6$$

We know that, A^{-1} exists, if A is non-singular matrix i.e., $|A| \neq 0$.

$$\therefore 2 + 5\lambda + 6 \neq 0$$

$$\Rightarrow 5\lambda \neq -8$$

$$\therefore \lambda \neq \frac{-8}{5}$$

So, A^{-1} exists if and only if $\lambda \neq \frac{-8}{5}$.

Q. 30 If $f(t) = \begin{bmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{bmatrix}$, then $\lim_{t \rightarrow 0} \frac{f(t)}{t^2}$ is equal to

(a) 0

(b) -1

(c) 2

(d) 3

Sol. (a) We have,

$$f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$$

Expanding along C_1 ,

$$\begin{aligned} &= \cos t (t^2 - 2t^2) - 2\sin t (t^2 - t) + \sin t (2t^2 - t) \\ &= -t^2 \cos t - (t^2 - t)2\sin t + (2t^2 - t)\sin t \\ &= -t^2 \cos t - t^2 \cdot 2\sin t + t \cdot 2\sin t + 2t^2 \sin t \\ &= -t^2 \cos t + 2t \sin t \end{aligned}$$

$$\begin{aligned} \therefore \lim_{t \rightarrow 0} \frac{f(t)}{t^2} &= \lim_{t \rightarrow 0} \frac{(-t^2 \cos t)}{t^2} + \lim_{t \rightarrow 0} \frac{2t \sin t}{t^2} \\ &= -\lim_{t \rightarrow 0} \cos t + 2 \cdot \lim_{t \rightarrow 0} \frac{\sin t}{t} \\ &= -1 + 1 \\ &= 0 \end{aligned}$$

$$\left[\because \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1 \text{ and } \cos 0 = 1 \right]$$

Q. 31 The maximum value of

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin \theta & 1 \\ 1 + \cos \theta & 1 & 1 \end{vmatrix} \text{ is (where, } \theta \text{ is real number)}$$

(a) $\frac{1}{2}$

(b) $\frac{\sqrt{3}}{2}$

(c) $\sqrt{2}$

(d) $\frac{2\sqrt{3}}{4}$

Sol. (a) Since,

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin \theta & 1 \\ 1 + \cos \theta & 1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 & 1 \\ 0 & \sin \theta & 1 \\ \cos \theta & 0 & 1 \end{vmatrix} \quad [\because C_1 \rightarrow C_1 - C_3 \text{ and } C_2 \rightarrow C_2 - C_3] \\ &= 1(\sin \theta \cdot \cos \theta) \\ &= \frac{1}{2} \cdot 2 \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta \end{aligned}$$

Since, the maximum value of $\sin 2\theta$ is 1. So, for maximum value of θ should be 45° .

$$\begin{aligned} \therefore \Delta &= \frac{1}{2} \sin 2 \cdot 45^\circ \\ &= \frac{1}{2} \sin 90^\circ = \frac{1}{2} \cdot 1 = \frac{1}{2} \end{aligned}$$

On taking $(2\cos x + \sin x)$ common from C_1 , we get

$$\Rightarrow (2\cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$$\Rightarrow (2\cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & (\sin x - \cos x) \end{vmatrix} = 0$$

$[\because R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$

Expanding along C_1 ,

$$(2\cos x + \sin x) [1 \cdot (\sin x - \cos x)^2] = 0$$

$$\Rightarrow (2\cos x + \sin x) (\sin x - \cos x)^2 = 0$$

$$\text{Either } 2\cos x = -\sin x$$

$$\Rightarrow \cos x = -\frac{1}{2}\sin x$$

$$\Rightarrow \tan x = -2 \quad \dots(i)$$

But here for $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$, we get $-1 \leq \tan x \leq 1$ so, no solution possible

and for $(\sin x - \cos x)^2 = 0$, $\sin x = \cos x$

$$\Rightarrow \tan x = 1 = \tan \frac{\pi}{4}$$

$$\therefore x = \frac{\pi}{4}$$

So, only one distinct real root exist.

Q. 29 If A , B and C are angles of a triangle, then the determinant

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} \text{ is equal to}$$

(a) 0

(b) -1

(c) 1

(d) None of these

Sol. (a) We have, $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$

Applying $C_1 \rightarrow aC_1 + bC_2 + cC_3$,

$$\begin{vmatrix} -a + b\cos C + c\cos B & \cos C & \cos B \\ a\cos C - b + c\cos A & -1 & \cos A \\ a\cos B + b\cos A - c & \cos A & -1 \end{vmatrix}$$

Also, by projection rule in a triangle, we know that

$$a = b\cos C + c\cos B, b = c\cos A + a\cos C \text{ and } c = a\cos B + b\cos A$$

Using above equation in column first, we get

$$\begin{vmatrix} -a + a & \cos C & \cos B \\ b - b & -1 & \cos A \\ c - c & \cos A & -1 \end{vmatrix} = \begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = 0$$

[since, determinant having all elements of any column or row gives value of determinant as zero]

Expanding along R_1 ,

$$9 = \frac{1}{2} [-3(-k) - 0 + 1(3k)]$$

$\Rightarrow$

$$18 = 3k + 3k = 6k$$

$\therefore$

$$k = \frac{18}{6} = 3$$

Q. 27 The determinant

$$\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix} \text{ equals to}$$

(a) $abc(b - c)(c - a)(a - b)$

(b) $(b - c)(c - a)(a - b)$

(c) $(a + b + c)(b - c)(c - a)(a - b)$

(d) None of these

Sol. (d) We have,

$$\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix} = \begin{vmatrix} b(b - a) & b - c & c(b - a) \\ a(b - a) & a - b & b(b - a) \\ c(b - a) & c - a & a(b - a) \end{vmatrix}$$

$$= (b - a)^2 \begin{vmatrix} b & b - c & c \\ a & a - b & b \\ c & c - a & a \end{vmatrix}$$

[on taking $(b - a)$ common from C_1 and C_3 each]

$$= (b - a)^2 \begin{vmatrix} b - c & b - c & c \\ a - b & a - b & b \\ c - a & c - a & a \end{vmatrix} \quad [\because C_1 \rightarrow C_1 - C_3]$$

$$= 0$$

[since, two columns C_1 and C_2 are identical, so the value of determinant is zero]

Q. 28 The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the

interval $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$ is

(a) 0

(b) 2

(c) 1

(d) 3

Sol. (c) We have,

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$,

$$\begin{vmatrix} 2\cos x + \sin x & \cos x & \cos x \\ 2\cos x + \sin x & \sin x & \cos x \\ 2\cos x + \sin x & \cos x & \sin x \end{vmatrix} = 0$$

Objective Type Questions

Q. 24 If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then the value of x is

(a) 3

(b) ± 3

(c) ± 6

(d) 6

Sol. (c) $\therefore \begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

$$\Rightarrow 2x^2 - 40 = 18 + 14$$

$$\Rightarrow 2x^2 = 32 + 40$$

$$\Rightarrow x^2 = \frac{72}{2} = 36$$

$$\therefore x = \pm 6$$

Q. 25 The value of $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$ is

(a) $a^3 + b^3 + c^3$

(b) $3bc$

(c) $a^3 + b^3 + c^3 - 3abc$

(d) None of these

Sol. (d) We have,

$$\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix} = \begin{vmatrix} a+c & b+c+a & a \\ b+c & c+a+b & b \\ c+b & a+b+c & c \end{vmatrix} \quad [\because C_1 \rightarrow C_1 + C_2 \text{ and } C_2 \rightarrow C_2 + C_3]$$

$$= (a+b+c) \begin{vmatrix} a+c & 1 & a \\ b+c & 1 & b \\ c+b & 1 & c \end{vmatrix} \quad [\text{taking } (a+b+c) \text{ common from } C_2]$$

$$= (a+b+c) \begin{vmatrix} a-b & 0 & a-c \\ 0 & 0 & b-c \\ c+b & 1 & c \end{vmatrix} \quad [\because R_2 \rightarrow R_2 - R_3 \text{ and } R_1 \rightarrow R_1 - R_3]$$

$$= (a+b+c) [-(b-c) \cdot (a-b)] \quad [\text{expanding along } R_2]$$

$$= (a+b+c)(c-b)(a-b)$$

Q. 26 If the area of a triangle with vertices $(-3, 0)$, $(3, 0)$ and $(0, k)$ is 9 sq units. Then, the value of k will be

(a) 9

(b) 3

(c) -9

(d) 6

Sol. (b) We know that, area of a triangle with vertices (x_1, y_1) , (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\therefore \Delta = \frac{1}{2} \begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}$$

Q. 23 If $x + y + z = 0$, then prove that $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

Thinking Process

We have, given $x + y + z = 0 \Rightarrow x^3 + y^3 + z^3 = 3xyz$. So, by using this in solving the given determinant from both the sides, we can equate the obtained result from both the sides to desired result.

Sol. Since, $x + y + z = 0$, also we have to prove

$$\begin{aligned} \therefore \quad \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} &= xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \\ \text{LHS} &= \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} \\ &= xa(za \cdot ya - xb \cdot xc) - yb(yc \cdot ya - xb \cdot zb) + zc(yc \cdot xc - za \cdot zb) \\ &= xa(a^2yz - x^2bc) - yb(y^2ac - b^2xz) + zc(c^2xy - z^2ab) \\ &= x y z a^3 - x^3 abc - y^3 abc + b^3 x y z + c^3 x y z - z^3 abc \\ &= x y z (a^3 + b^3 + c^3) - abc (x^3 + y^3 + z^3) \\ &= x y z (a^3 + b^3 + c^3) - abc (3 x y z) \quad [\because x + y + z = 0 \Rightarrow x^3 + y^3 + z^3 - 3xyz] \\ &= x y z (a^3 + b^3 + c^3 - 3abc) \quad \dots(i) \end{aligned}$$

$$\begin{aligned} \text{Now,} \quad \text{RHS} &= xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = xyz \begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix} \quad [\because C_1 \rightarrow C_1 + C_2 + C_3] \\ &= xyz(a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix} \quad [\text{taking } (a+b+c) \text{ common from } C_1] \\ &= xyz(a+b+c) \begin{vmatrix} 0 & b-c & c-a \\ 0 & a-c & b-a \\ 1 & c & a \end{vmatrix} \\ &\quad [\because R_1 \rightarrow R_1 - R_3 \text{ and } R_2 \rightarrow R_2 - R_3] \end{aligned}$$

Expanding along C_1 ,

$$\begin{aligned} &= xyz(a+b+c) [1(b-c)(b-a) - (a-c)(c-a)] \\ &= xyz(a+b+c) (b^2 - ab - bc + ac + a^2 + c^2 - 2ac) \\ &= xyz(a+b+c) (a^2 + b^2 + c^2 - ab - bc - ca) \\ &= xyz(a^3 + b^3 + c^3 - 3abc) \quad \dots(ii) \end{aligned}$$

From Eqs. (i) and (ii),

$$\Rightarrow \quad \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Hence proved.

Q. 22 Prove that $\begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ca - b^2 \end{vmatrix}$ is divisible by $(a + b + c)$ and

find the quotient.

Sol. Let $\Delta = \begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ca - b^2 \end{vmatrix}$

$$= \begin{vmatrix} bc - a^2 - ca + b^2 & ca - b^2 - ab + c^2 & ab - c^2 \\ ca - b^2 - ab + c^2 & ab - c^2 - bc + a^2 & bc - a^2 \\ ab - c^2 - bc + a^2 & bc - a^2 - ca + b^2 & ca - b^2 \end{vmatrix}$$

[$\because C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$]

$$= \begin{vmatrix} (b-a)(a+b+c) & (c-b)(a+b+c) & ab - c^2 \\ (c-b)(a+b+c) & (a-c)(a+b+c) & bc - a^2 \\ (a-c)(a+b+c) & (b-a)(a+b+c) & ca - b^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b-a & c-b & ab - c^2 \\ c-b & a-c & bc - a^2 \\ a-c & b-a & ca - b^2 \end{vmatrix}$$

[taking $(a+b+c)$ common from C_1 and C_2 each]

$$= (a+b+c)^2 \begin{vmatrix} 0 & 0 & ab + bc + ca - (a^2 + b^2 + c^2) \\ c-b & a-c & bc - a^2 \\ a-c & b-a & ca - b^2 \end{vmatrix}$$

[$\because R_1 \rightarrow R_1 + R_2 + R_3$]

Now, expanding along R_1 ,

$$= (a+b+c)^2 [ab + bc + ca - (a^2 + b^2 + c^2)](c-b)(b-a) - (a-c)^2$$

$$= (a+b+c)^2 (ab + bc + ca - a^2 - b^2 - c^2)$$

$$(cb - ac - b^2 + ab - a^2 - c^2 + 2ac)$$

$$= (a+b+c)^2 (a^2 + b^2 + c^2 - ab - bc - ca)$$

$$(a^2 + b^2 + c^2 - ac - ab - bc)$$

$$= \frac{1}{2} (a+b+c) [(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)]$$

$$[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

$$= \frac{1}{2} (a+b+c)(a^3 + b^3 + c^3 - 3abc)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

Hence, given determinant is divisible by $(a + b + c)$ and quotient is

$$(a^3 + b^3 + c^3 - 3abc)[(a-b)^2 + (b-c)^2 + (c-a)^2].$$

$$\begin{aligned}
 \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} \\
 &= \frac{1}{6} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} \quad [\text{using Eq. (i)}] \\
 &= \frac{1}{6} \begin{bmatrix} 6 + 34 - 28 \\ -12 + 34 - 28 \\ 6 - 17 + 35 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \\
 \therefore x &= 2, y = -1 \text{ and } z = 4
 \end{aligned}$$

Q. 21 If $a + b + c \neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then prove that $a = b = c$.

Sol. Let $A = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$\begin{aligned}
 &= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} \quad [\because R_1 \rightarrow R_1 + R_2 + R_3] \\
 &= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} \\
 &= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ b-a & c-a & a \\ c-b & a-b & b \end{vmatrix} \quad [\because C_1 \rightarrow C_1 - C_3 \text{ and } C_2 \rightarrow C_2 - C_3]
 \end{aligned}$$

Expanding along R_1 ,

$$\begin{aligned}
 &= (a+b+c) [1(b-a)(a-b) - (c-a)(c-b)] \\
 &= (a+b+c) (ba - b^2 - a^2 + ab - c^2 + cb + ac - ab) \\
 &= \frac{-1}{2} (a+b+c) \times (-2) (-a^2 - b^2 - c^2 + ab + bc + ca) \\
 &= \frac{-1}{2} (a+b+c) [a^2 + b^2 + c^2 - 2ab - 2bc - 2ca + a^2 + b^2 + c^2] \\
 &= -\frac{1}{2} (a+b+c) [a^2 + b^2 - 2ab + b^2 + c^2 - 2bc + c^2 + a^2 - 2ac] \\
 &= \frac{-1}{2} (a+b+c) [(a-b)^2 + (b-c)^2 + (c-a)^2]
 \end{aligned}$$

Also,

$$\begin{aligned}
 A &= 0 \\
 &= \frac{-1}{2} (a+b+c) [(a-b)^2 + (b-c)^2 + (c-a)^2] = 0 \\
 (a-b)^2 + (b-c)^2 + (c-a)^2 &= 0 \quad [\because a+b+c \neq 0, \text{ given}]
 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
 a-b &= b-c = c-a = 0 \\
 a &= b = c
 \end{aligned}$$

Hence proved.

In the form of $AX = B$,

$$\begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

For A^{-1} ,

$$|A| = |3(5) - 2(1 - 6) + (-2)(-5)| \\ = |15 + 10 + 10| = |35| \neq 0$$

$$\therefore A_{11} = 5, A_{12} = 5, A_{13} = -5, A_{21} = 0, A_{22} = 7, A_{23} = 7, A_{31} = 10, A_{32} = -11 \text{ and } A_{33} = 4$$

$$\therefore \text{adj } A = \begin{bmatrix} 5 & 5 & -5 \\ 0 & 7 & 7 \\ 10 & -11 & 4 \end{bmatrix}^T = \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{35} \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

For $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \\ = \frac{1}{35} \begin{bmatrix} 15 + 20 \\ 15 + 42 - 22 \\ -15 + 42 + 8 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 35 \\ 35 \\ 35 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore x = 1, y = 1 \text{ and } z = 1$$

Q. 20 If $A = \begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix}$ and $B = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$, then find BA and use this to solve the system of equations $y + 2z = 7$, $x - y = 3$ and $2x + 3y + 4z = 17$.

Sol. We have,

$$A = \begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix} \text{ and } B = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$$

$$\therefore BA = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix} \begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 6I$$

$$\therefore B^{-1} = \frac{A}{6} = \frac{1}{6}A = \frac{1}{6} \begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix} \quad \dots(i)$$

Also, $x - y = 3, 2x + 3y + 4z = 17$ and $y + 2z = 7$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$\begin{aligned} \therefore A^{-1} &= \frac{\text{adj } A}{|A|} \\ &= \frac{1}{1} \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \\ \Rightarrow A^{-1} &= \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad \dots(ii) \end{aligned}$$

Also, we have the system of linear equations as

$$x - 2y = 10,$$

$$2x - y - z = 8$$

and

$$-2y + z = 7$$

In the form of $CX = D$,

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

where, $C = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$

We know that,

$$(A^T)^{-1} = (A^{-1})^T$$

$$\therefore C^T = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix} = A \quad \text{[using Eq. (i)]}$$

$$\therefore X = C^{-1} D$$

$$\begin{aligned} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} -30 + 16 + 14 \\ -20 + 8 + 7 \\ -40 + 16 + 21 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix} \end{aligned}$$

$$\therefore x = 0, y = -5 \text{ and } z = -3$$

Q. 19 Using matrix method, solve the system of equations $3x + 2y - 2z = 3$,
 $x + 2y + 3z = 6$ and $2x - y + z = 2$.

Thinking Process

We know that, for given system of equations in the matrix form, we get $AX = B \Rightarrow X = A^{-1}B$,

where $A^{-1} = \frac{\text{adj}(A)}{|A|}$ and then by getting inverse of A and determinant of A , we can get

the desired result.

Sol. Given system of equations is

$$3x + 2y - 2z = 3,$$

$$x + 2y + 3z = 6$$

and

$$2x - y + z = 2$$

Q. 17 Find A^{-1} , if $A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ and show that $A^{-1} = \frac{A^2 - 3I}{2}$.

Sol. We have, $A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$$\therefore A_{11} = -1, A_{12} = 1, A_{13} = 1, A_{21} = 1, A_{22} = -1, A_{23} = 1, A_{31} = 1, A_{32} = 1 \text{ and } A_{33} = -1$$

$$\therefore \text{adj } A = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}^T = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$\text{and } |A| = -1(-1) + 1 \cdot 1 = 2$$

$$\therefore A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad \dots(i)$$

$$\text{and } A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \dots(ii)$$

$$\therefore \frac{A^2 - 3I}{2} = \frac{1}{2} \left\{ \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} \right\} = \frac{1}{2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= A^{-1}$$

[using Eq. (i)]
Hence proved.

Long Answer Type Questions

Q. 18 If $A = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix}$, then find the value of A^{-1} .

Using A^{-1} , solve the system of linear equations $x - 2y = 10$, $2x - y - z = 8$ and $-2y + z = 7$.

Sol. We have, $A = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \quad \dots(i)$

$$\therefore |A| = 1(-3) - 2(-2) + 0 = 1 \neq 0$$

$$\text{Now, } A_{11} = -3, A_{12} = 2, A_{13} = 2, A_{21} = -2, A_{22} = 1, A_{23} = 1, A_{31} = -4, A_{32} = 2 \text{ and } A_{33} = 3$$

$$\therefore \text{adj } (A) = \begin{vmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{vmatrix}^T = \begin{vmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

Q. 15 Show that the points $(a + 5, a - 4)$, $(a - 2, a + 3)$ and (a, a) do not lie on a straight line for any value of a .

Thinking Process

We know that, if three points lie in a straight line, then area formed by these points will be equal to zero. So, by showing area formed by these points other than zero, we can prove the result.

Sol. Given, the points are $(a + 5, a - 4)$, $(a - 2, a + 3)$ and (a, a) .

$$\begin{aligned} \therefore \Delta &= \frac{1}{2} \begin{vmatrix} a+5 & a-4 & 1 \\ a-2 & a+3 & 1 \\ a & a & 1 \end{vmatrix} \\ &= \frac{1}{2} \begin{vmatrix} 5 & -4 & 0 \\ -2 & 3 & 0 \\ a & a & 1 \end{vmatrix} \quad [\because R_1 \rightarrow R_1 - R_3 \text{ and } R_2 \rightarrow R_2 - R_3] \\ &= \frac{1}{2} [1(15 - 8)] \\ \Rightarrow &= \frac{7}{2} \neq 0 \end{aligned}$$

Hence, given points form a triangle i.e., points do not lie in a straight line.

Q. 16 Show that $\triangle ABC$ is an isosceles triangle, if the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0.$$

Sol. We have, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos C & \cos B - \cos C & 1 + \cos C \\ \cos^2 A + \cos A - \cos^2 C - \cos C & \cos^2 B + \cos B - \cos^2 C - \cos C & \cos^2 C + \cos C \end{vmatrix} = 0$$

[$\because C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$]

$$\Rightarrow (\cos A - \cos C) \cdot (\cos B - \cos C)$$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 + \cos C \\ \cos A + \cos C + 1 & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix} = 0$$

[taking $(\cos A - \cos C)$ common from C_1 and $(\cos B - \cos C)$ common from C_2]

$$\Rightarrow (\cos A - \cos C) \cdot (\cos B - \cos C) [(\cos B + \cos C + 1) - (\cos A + \cos C + 1)] = 0$$

$$\Rightarrow (\cos A - \cos C) \cdot (\cos B - \cos C) (\cos B + \cos C + 1 - \cos A - \cos C - 1) = 0$$

$$\Rightarrow (\cos A - \cos C) \cdot (\cos B - \cos C) (\cos B - \cos A) = 0$$

$$\text{i.e.,} \quad \cos A = \cos C \text{ or } \cos B = \cos C \text{ or } \cos B = \cos A$$

$$\Rightarrow A = C \text{ or } B = C \text{ or } B = A$$

Hence, ABC is an isosceles triangle.

Q. 13 If $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$, then find the value of x .

Sol. Given, $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 12+x & 12+x & 12+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0 \quad [\because R_1 \rightarrow R_1 + R_2 + R_3]$$

$$\Rightarrow (12+x) \begin{vmatrix} 1 & 1 & 1 \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0 \quad [\text{taking } (12+x) \text{ common from } R_1]$$

$$\Rightarrow (12+x) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 8 & 4+x \\ 2x & 8 & 4-x \end{vmatrix} = 0 \quad [\because C_1 \rightarrow C_1 - C_3 \text{ and } C_2 \rightarrow C_2 + C_3]$$

$$\Rightarrow (12+x) [1 \cdot (-16x)] = 0$$

$$\Rightarrow (12+x)(-16x) = 0$$

$$\therefore x = -12, 0$$

Q. 14 If $a_1, a_2, a_3, \dots, a_r$ are in GP, then prove that the determinant

$$\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix} \text{ is independent of } r.$$

💡 Thinking Process

We know that, n th term of a GP has value ar^{n-1} , where a = first term and r = common ratio. So, by using this result, we can prove the given determinant as independent of r .

Sol. We know that, $a_{r+1} = AR^{(r+1)-1} = AR^r$

where $r = r$ th term of a GP, A = First term of a GP and R = Common ratio of GP

We have, $\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$

$$= \begin{vmatrix} AR^r & AR^{r+4} & AR^{r+8} \\ AR^{r+6} & AR^{r+10} & AR^{r+14} \\ AR^{r+10} & AR^{r+16} & AR^{r+20} \end{vmatrix}$$

$$= AR^r \cdot AR^{r+6} \cdot AR^{r+10} \begin{vmatrix} 1 & AR^4 & AR^8 \\ 1 & AR^4 & AR^8 \\ 1 & AR^6 & AR^{10} \end{vmatrix}$$

[taking AR^r , AR^{r+6} and AR^{r+10} common from R_1 , R_2 and R_3 , respectively]

$= 0$ [since, R_1 and R_2 are identicals]

From Eqs. (i) and (ii),
$$\frac{3}{16} a^4 = \frac{1}{4} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$$

$$\Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3}{4} a^4$$

Hence proved.

Q. 12 Find the value of θ satisfying
$$\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$$

Sol. We have,

$$\begin{vmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & \sin 3\theta \\ -7 & 3 & \cos 2\theta \\ 14 & -7 & -2 \end{vmatrix} = 0 \quad [\because C_1 \rightarrow C_1 - C_2]$$

$$\Rightarrow 7 \begin{vmatrix} 0 & 1 & \sin 3\theta \\ -1 & 3 & \cos 2\theta \\ 2 & -7 & -2 \end{vmatrix} = 0 \quad [\text{taking 7 common from } C_1]$$

$$\Rightarrow 7 [0 - 1(2 - 2\cos 2\theta) + \sin 3\theta(7 - 6)] = 0 \quad [\text{expanding along } R_1]$$

$$\Rightarrow 7 [-2(1 - \cos 2\theta) + \sin 3\theta] = 0$$

$$\Rightarrow -14 + 14\cos 2\theta + 7\sin 3\theta = 0$$

$$\Rightarrow 14\cos 2\theta + 7\sin 3\theta = 14$$

$$\Rightarrow 14(1 - 2\sin^2 \theta) + 7(3\sin \theta - 4\sin^3 \theta) = 14$$

$$\Rightarrow -28\sin^2 \theta + 14 + 21\sin \theta - 28\sin^3 \theta = 14$$

$$\Rightarrow -28\sin^2 \theta - 28\sin^3 \theta + 21\sin \theta = 0$$

$$\Rightarrow 28\sin^3 \theta + 28\sin^2 \theta - 21\sin \theta = 0$$

$$\Rightarrow 4\sin^3 \theta + 4\sin^2 \theta - 3\sin \theta = 0$$

$$\Rightarrow \sin \theta(4\sin^2 \theta + 4\sin \theta - 3) = 0$$

$$\Rightarrow \text{Either } \sin \theta = 0,$$

$$\Rightarrow \theta = n\pi \text{ or } 4\sin^2 \theta + 4\sin \theta - 3 = 0$$

$$\therefore \sin \theta = \frac{-4 \pm \sqrt{16 + 48}}{8} = \frac{-4 \pm \sqrt{64}}{8}$$

$$= \frac{-4 \pm 8}{8} = \frac{4}{8}, \frac{-12}{8}$$

$$\sin \theta = \frac{1}{2}, \frac{-3}{2}$$

If $\sin \theta = \frac{1}{2} = \sin \frac{\pi}{6}$, then

$$\theta = n\pi + (-1)^n \frac{\pi}{6}$$

Hence, $\sin \theta = \frac{-3}{2}$

[not possible because $-1 \leq \sin \theta \leq 1$]

Q. 10 If $A + B + C = 0$, then prove that
$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0.$$

Thinking Process

We have, given $A + B + C = 0$, so on solving the determinant by expansion, we can use $\cos(A + B) = \cos(-C)$ and similarly after simplification this expansion we will get the desired result.

Sol. We have to prove,
$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$$

$$\begin{aligned} \therefore \text{LHS} &= \begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} \\ &= 1(1 - \cos^2 A) - \cos C(\cos C - \cos A \cdot \cos B) + \cos B(\cos C \cdot \cos A - \cos B) \\ &= \sin^2 A - \cos^2 C + \cos A \cdot \cos B \cdot \cos C + \cos A \cdot \cos B \cdot \cos C - \cos^2 B \\ &= \sin^2 A - \cos^2 B + 2 \cos A \cdot \cos B \cdot \cos C - \cos^2 C \\ &= -\cos(A + B) \cdot \cos(A - B) + 2 \cos A \cdot \cos B \cdot \cos C - \cos^2 C \\ &\quad [\because \cos^2 B - \sin^2 A = \cos(A + B) \cdot \cos(A - B)] \\ &= -\cos(-C) \cdot \cos(A - B) + \cos C(2 \cos A \cdot \cos B - \cos C) \quad [\because \cos(-\theta) = \cos \theta] \\ &= -\cos C(\cos A \cdot \cos B + \sin A \cdot \sin B - 2 \cos A \cdot \cos B + \cos C) \\ &= \cos C(\cos A \cdot \cos B - \sin A \cdot \sin B - \cos C) \\ &= \cos C[\cos(A + B) - \cos C] \\ &= \cos C(\cos C - \cos C) = 0 = \text{RHS} \end{aligned}$$

Hence proved.

Q. 11 If the coordinates of the vertices of an equilateral triangle with sides of length ' a ' are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}.$$

Sol. Since, we know that area of a triangle with vertices (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , is given by

$$\begin{aligned} \Delta &= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \\ \Rightarrow \Delta^2 &= \frac{1}{4} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 \quad \dots(i) \end{aligned}$$

We know that, area of an equilateral triangle with side a ,

$$\begin{aligned} \Delta &= \frac{1}{2} \left(\frac{\sqrt{3}}{2} \right) a^2 = \frac{\sqrt{3}}{4} a^2 \\ \Rightarrow \Delta^2 &= \frac{3}{16} a^4 \quad \dots(ii) \end{aligned}$$

$$\begin{aligned}
\therefore \text{LHS} &= \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} \\
&= \begin{vmatrix} y+z+z+y & z & y \\ z+z+x+x & z+x & x \\ y+x+x+y & x & x+y \end{vmatrix} & [\because C_1 \rightarrow C_1 + C_2 + C_3] \\
&= 2 \begin{vmatrix} (y+z) & z & y \\ (z+x) & z+x & x \\ (x+y) & x & x+y \end{vmatrix} & [\text{taking 2 common from } C_1] \\
&= 2 \begin{vmatrix} y & z & y \\ 0 & z+x & x \\ y & x & x+y \end{vmatrix} & [\because C_1 \rightarrow C_1 - C_2] \\
&= 2 \begin{vmatrix} 0 & z-x & -x \\ 0 & z+x & x \\ y & x & x+y \end{vmatrix} & [\because R_1 \rightarrow R_1 - R_3] \\
&= 2 [y(xz - x^2 + xz + x^2)] \\
&= 4xyz = \text{RHS} & \text{Hence proved.}
\end{aligned}$$

Q. 9 $\begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a - 1)^3$

💡 Thinking Process

Here, by using $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$ in LHS, we can easily get the desired result.

Sol. We have to prove,

$$\begin{aligned}
&= \begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a - 1)^3 \\
\therefore \text{LHS} &= \begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} \\
&= \begin{vmatrix} a^2 + 2a - 2a - 1 & 2a + 1 - a - 2 & 0 \\ 2a + 1 - 3 & a + 2 - 3 & 0 \\ 3 & 3 & 1 \end{vmatrix} & [\because R_1 \rightarrow R_1 - R_2 \text{ and } R_2 \rightarrow R_2 - R_3] \\
&= \begin{vmatrix} (a - 1)(a + 1) & (a - 1) & 0 \\ 2(a - 1) & (a - 1) & 0 \\ 3 & 3 & 1 \end{vmatrix} = (a - 1)^2 \begin{vmatrix} (a + 1) & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix} \\
& & [\text{taking } (a - 1) \text{ common from } R_1 \text{ and } R_2 \text{ each}] \\
&= (a - 1)^2 [1(a + 1) - 2] = (a - 1)^3 \\
&= \text{RHS} & \text{Hence proved.}
\end{aligned}$$

Expanding along R_1 ,

$$\begin{aligned} &= (a + b + c) [1\{0 + (a + b + c)^2\}] \\ &= (a + b + c) [(a + b + c)^2] \\ &= (a + b + c)^3 \end{aligned}$$

Q. 7
$$\begin{vmatrix} y^2 z^2 & yz & y + z \\ z^2 x^2 & zx & z + x \\ x^2 y^2 & xy & x + y \end{vmatrix} = 0$$

Sol. We have to prove,

$$\begin{aligned} &\begin{vmatrix} y^2 z^2 & yz & y + z \\ z^2 x^2 & zx & z + x \\ x^2 y^2 & xy & x + y \end{vmatrix} = 0 \\ \therefore \text{LHS} &= \begin{vmatrix} y^2 z^2 & yz & y + z \\ z^2 x^2 & zx & z + x \\ x^2 y^2 & xy & x + y \end{vmatrix} = \frac{1}{xyz} \begin{vmatrix} x y^2 z^2 & x yz & x y + x z \\ x^2 y z^2 & x yz & yz + x y \\ x^2 y^2 z & x yz & x z + yz \end{vmatrix} \\ &\quad [\because R_1 \rightarrow x R_1, R_2 \rightarrow y R_2, R_3 \rightarrow z R_3] \\ &= \frac{1}{xyz} (xyz)^2 \begin{vmatrix} yz & 1 & xy + xz \\ xz & 1 & yz + xy \\ xy & 1 & xz + yz \end{vmatrix} \\ &\quad [\text{taking } (xyz) \text{ common from } C_1 \text{ and } C_2] \\ &= xyz \begin{vmatrix} yz & 1 & xy + yz + zx \\ xz & 1 & xy + yz + zx \\ xy & 1 & xy + yz + zx \end{vmatrix} [C_3 \rightarrow C_3 + C_1] \\ &= xyz (xy + yz + zx) \begin{vmatrix} yz & 1 & 1 \\ xz & 1 & 1 \\ xy & 1 & 1 \end{vmatrix} \\ &\quad [\text{taking } (xy + yz + zx) \text{ common from } C_3] \\ &= 0 \quad [\text{since, } C_2 \text{ and } C_3 \text{ are identicals}] \\ &= \text{RHS} \end{aligned}$$

Hence proved.

Q. 8
$$\begin{vmatrix} y + z & z & y \\ z & z + x & x \\ y & x & x + y \end{vmatrix} = 4xyz$$

💡 Thinking Process

First in LHS use $C_1 \rightarrow C_1 + C_2 + C_3$ and then by using $C_1 \rightarrow C_1 - C_2$ and $R_1 \rightarrow R_1 - R_3$, we can get two zeroes in column 1 and then by simplification we will get the desired result.

Sol. We have to prove,

$$\begin{vmatrix} y + z & z & y \\ z & z + x & x \\ y & x & x + y \end{vmatrix} = 4xyz$$

Q. 5
$$\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

Sol. We have,
$$\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = \begin{vmatrix} 2x+4 & 2x+4 & 2x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} \quad [\because R_1 \rightarrow R_1 + R_2]$$

$$= \begin{vmatrix} 2x & 2x & 2x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} + \begin{vmatrix} 4 & 4 & 0 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

[here, given determinant is expressed in sum of two determinants]

$$= 2x \begin{vmatrix} 1 & 1 & 1 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} + 4 \begin{vmatrix} 1 & 1 & 0 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

[taking 2x common from first row of first determinant and 4 from first row of second determinant]

Applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$ in first and applying $C_1 \rightarrow C_1 - C_2$ in second, we get

$$= 2x \begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & x \\ -4 & -4 & x+4 \end{vmatrix} + 4 \begin{vmatrix} 0 & 1 & 0 \\ -4 & x+4 & x \\ 0 & x & x+4 \end{vmatrix}$$

Expanding both the along first column, we get

$$\begin{aligned} & 2x [-4(-4)] + 4 [4(x+4) - 0] \\ &= 2x \times 16 + 16(x+4) \\ &= 32x + 16x + 64 \\ &= 16(3x+4) \end{aligned}$$

Q. 6
$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Sol. We have,
$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad [\because R_1 \rightarrow R_1 + R_2 + R_3]$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

[taking (a+b+c) common from the first row]

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & -(a+b+c) & 2b \\ (a+b+c) & (a+b+c) & (c-a-b) \end{vmatrix}$$

$[\because C_1 \rightarrow C_1 - C_3 \text{ and } C_2 \rightarrow C_2 - C_3]$

Q. 3
$$\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$$

Sol. We have,
$$\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix} = x^2y^2z^2 \begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix}$$

 [taking x^2, y^2 and z^2 common from C_1, C_2 and C_3 , respectively]

$$= x^2y^2z^2 \begin{vmatrix} 0 & 0 & x \\ y & -y & y \\ z & z & 0 \end{vmatrix} \quad [\because C_2 \rightarrow C_2 - C_3]$$

$$= x^2y^2z^2 [x(yz + yz)]$$

$$= x^2y^2z^2 \cdot 2xyz = 2x^3y^3z^3$$

Q. 4
$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

Sol. We have,
$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

Applying, $C_1 \rightarrow C_1 + C_2 + C_3$,

$$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

$$= (x+y+z) \begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

[taking $(x+y+z)$ common from column C_1]

$$= (x+y+z) \begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & x-y \\ 0 & x-z & 2z+x \end{vmatrix}$$

[$\because R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$]

Now, expanding along first column, we get

$$(x+y+z) \cdot 1 [(2y+x)(2z+x) - (x-y)(x-z)]$$

$$= (x+y+z) (4yz + 2yx + 2xz + x^2 - x^2 + xz + yx - yz)$$

$$= (x+y+z) (3yz + 3yx + 3xz)$$

$$= 3(x+y+z)(yz + yx + xz)$$

Determinants

Short Answer Type Questions

Q. 1 $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Sol. We have, $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = \begin{vmatrix} x^2 - 2x + 2 & x - 1 \\ 0 & x + 1 \end{vmatrix} \quad [\because C_1 \rightarrow C_1 - C_2]$

$$= (x^2 - 2x + 2) \cdot (x + 1) - (x - 1) \cdot 0$$

$$= x^3 - 2x^2 + 2x + x^2 - 2x + 2$$

$$= x^3 - x^2 + 2$$

Q. 2 $\begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix}$

Sol. We have, $\begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix} = \begin{vmatrix} a & -a & 0 \\ 0 & a & -a \\ x & y & a + z \end{vmatrix} \quad \left[\begin{array}{l} \because R_1 \rightarrow R_1 - R_2 \\ \text{and } R_2 \rightarrow R_2 - R_3 \end{array} \right]$

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & a & -a \\ x & x + y & a + z \end{vmatrix} \quad [\because C_2 \rightarrow C_2 + C_1]$$

$$= a(a^2 + az + ax + ay)$$

$$= a^2(a + z + x + y)$$